# Theory of degenerate Bose gas without anomalous averages

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Theory of a weakly non-ideal Bose gas in the canonical ensemble is developed without assumption of the C-number representation of the creation and annihilation operators with zero momentum. It is shown that the pole of the "density-density" Green's function exactly coincides with the Bogolybov's phonon-roton spectrum of excitations. At the same time, a gap exists in the one-particle excitation spectrum . This gap is related to the density of particles in the "condensate".

PACS number(s): 05.30.Jp, 03.75.Kk, 03.75.Nt, 05.70.Fh



## I. INTRODUCTION

Starting with the Bogolyubov's papers [1,2], the microscopic theory of the degenerate Bose gas has been based on the special assumption that the creation $a_0^+$ and annihilation $a_0$ operators of particles with zero momentum can be replaced by a C-number

$$a_0^+ = a_0 = (< N_0 >)^{1/2}, \tag{1}$$

where $< N_0 >$ is the average quantity of particles in the state with $\mathbf{p} = 0$ ("condensate").

This assumption leads to the necessity of introducing in the theory the *anomalous averages* ("quasi-averages"), which is unnatural for the homogeneous and isotropic system under consideration. Let us pay attention that statement (1) in Bogolyubov's papers is directly

related to the requirement of the weak interparticle interaction in the degenerated Bose gas. Though the Bogolyubov theory (see, e.g., [3,4]) gives rise to many important and widely recognized results, such as the expression for the spectrum of excitations, explanation of the experimental data and agreement with the Landau superfluidity condition, there are serious doubts on the validity of the relation as well as on the agreement of the Hamiltonian corresponding to the assumption (1) with the original Hamiltonian of the system under consideration [5,6].

In this work, we propose a selfconsistent noncontradictory description of the degenerate Bose gas in which we do not use the assumption (1). In this way we show, in particular, that there exists the gap in the spectrum of one-particle excitations. The statement that the spectrum of one-particle excitations has a gap in parallel with the usual phonon-roton branch of the collective excitations in the degenerate Bose gas has been formulated in [7]. Later, the existence of the gap has been suggested in [8]. Spectra, thermodynamics, and dynamical structure factor for weakly non-ideal Bose gas below the condensation temperature $T < T_0$ have been considered in [9-11] in terms of the dielectric formalism, similar to that in plasma-like systems. The dielectric formalism based on the Bogolyubov assumption (1) for the operators $a_0^+$ and $a_0$ was developed in [12]. Recently, the gap-existence problem for the weakly non-ideal Bose gas has been considered by different methods in papers [13-16]. In fact, the paper [14] confirmed (independently) and developed not only the statement of Ref. [7] about the existence of a one-particle gap in the weakly non-ideal Bose gas below the condensation temperature, but also the results of the papers [9-11] on the possibility of considering the superfluid system without symmetry breaking. Surely, the existence of the gap disagrees with the conventional opinion that the gap is missing [3]. But we do believe that the conclusion about a missing gap is related to the Bogolubov simplifying assumption (1), and is not obliged to persist in a more rigorous theory.

It is necessary to stress that in the previous papers both Landau and Bogolyubov admitted that the one-particle excitation spectrum can have a gap, but later they dropped this idea, because of contradiction with observations of the phonon-roton branch of spectrum at small $q$ in the neutron scattering experiments. The approach which includes the both spectra, with and without a gap, has not been found and coexistence of these two branches of excitations was not suggested.

In the present work, we suggest that in the limit of strong degeneracy $T \to 0$, where $T$ is

the system temperature, all the particles tend to occupy the zero-momentum state, which means that

$$\lim_{T\to 0} < N_0 >=< N >, \tag{2}$$

where $< N >$ is the average number of particles in the system. This statement is confirmed in the paper by the self-consistent consideration. Below we use the canonical ensemble where $N$ is a given C-number.

## II. STATISTICAL SUM AND AVERAGES IN CANONICAL ENSEMBLE

Let us consider the Hamiltonian of the non-ideal Bose gas of particles with zero spin and a mass $m$ in a volume $V$

$$H = \sum_p \varepsilon_{\mathbf{p}}\, a^+_{\mathbf{p}} a_{\mathbf{p}} + \frac{1}{2V} \sum_{\mathbf{q},\mathbf{p_1},\mathbf{p_2}} u(q) a^+_{\mathbf{p_1-q/2}} a^+_{\mathbf{p_2+q/2}} a_{\mathbf{p_2-q/2}} a_{\mathbf{p_1+q/2}}, \tag{3}$$

where $a^+_p$ and $a_p$ are the creation and annihilation operators of particles with the momentum $\hbar\mathbf{p}$,

$$[a_{\mathbf{p_2}}, a^+_{\mathbf{p_1}}] = a_{\mathbf{p_2}} a^+_{\mathbf{p_1}} - a^+_{\mathbf{p_1}} a_{\mathbf{p_2}} = \delta_{\mathbf{p_1},\mathbf{p_2}}, \tag{4}$$

$\varepsilon_{\mathbf{p}} = \hbar^2 \mathbf{p}^2/2m$ is the energy spectrum of a free particle and $u(q)$ is the Fourier-component of the inter-particle interaction potential.

It is convenient to extract the particular term $U_0$ with $\mathbf{q} = 0$ from the sum over $\mathbf{q}$ in the Hamiltonian (3). This term can be written as

$$U_0 = u(0)\frac{N(N-1)}{2V}, \;\; N = \sum_p a^+_{\mathbf{p}} a_{\mathbf{p}}, \quad u(0) = u(q=0) = u(q \to 0), \tag{5}$$

where $N$ is the operator of the total number of particles, and $N$ is the C-number $N =< N >$. Hereafter, the brackets $< ... >$ mean the canonical-ensemble averaging. Since $U_0$ is also the C-number, one can write the statistical sum of the system under consideration in the form

$$Z = Sp\, \exp(-H/T) = \exp\left(-u(0)\frac{N(N-1)}{2VT}\right) Sp\, \exp(-H_0/T), \tag{6}$$

where

$$H_0 = H - U_0 = \sum_p \varepsilon_p\, a^+_{\mathbf{p}} a_{\mathbf{p}} + \frac{1}{2V} \sum_{\mathbf{q}\neq\mathbf{0},\mathbf{p_1},\mathbf{p_2}} u(q) a^+_{\mathbf{p_1-q/2}} a^+_{\mathbf{p_2+q/2}} a_{\mathbf{p_2-q/2}} a_{\mathbf{p_1+q/2}}, \tag{7}$$

Therefore, to provide convergence of the statistical sum (7) in the thermodynamic limit $V \to \infty$ $N \to \infty$, $n = N/V = const.$, the known condition $u(0) > 0$ has to be fulfilled. As $U_0$ is a C-number, it does not affect any averaging at all, and for calculating of average values the Hamiltonian $H$ is equivalent to $H_0$. For an arbitrary operator $< A >$, we get

$$< A >= Z^{-1} Sp \left\{ \exp(-H/T) A \right\} = Z_0^{-1} Sp \left\{ \exp(-H_0/T) A \right\} \equiv < A >_0; \quad Z_0 = Sp \, \exp(-H_0/T) \tag{8}$$

Let us now consider a more complicated situation associated with calculation of a time dependent correlation functions $f(t)$ such as

$$f(t) =< [A(t), B(0)] >, \quad A(t) = \exp(iHt/\hbar) A \exp(-iHt/\hbar). \tag{9}$$

The time dependence of the operators $a_p^+(t)$ and $a_p(t)$ can be represented in the form

$$a_p^+(t) = \exp(iH_0 t/\hbar) a_p^+ \exp(-iH_0 t/\hbar) \exp(iNu(0)t/V\hbar). \tag{10}$$

$$a_p(t) = \exp(-iNu(0)t/V\hbar) \exp(iH_0 t/\hbar) a_p \exp(-iH_0 t/\hbar). \tag{11}$$

Therefore, if in each of the operators $A$ and $B$ the numbers of creation and annihilation operators coincide (which is typical of the operators of the physical variables), the time-dependent correlation function can be written as

$$f(t) =< [A(t), B(0)] >_0, \quad A(t) = \exp(iH_0 t/\hbar) A \exp(-iH_0 t/\hbar). \tag{12}$$

On this basis, in what follows, we consider the average values with the Hamiltonian $H_0$ (7) within the canonical ensemble. The free energy of the initial system with the Hamiltonian $H$ (3), according to (5)-(7), reads

$$F = -T \ln Z = U_0 + F_0, \quad F_0 = -T \ln Z_0. \tag{13}$$

## III. EQUATIONS FOR "DENSITY-DENSITY" GREEN FUNCTION

Experimentally, the spectrum of collective excitations is found usually from data on the well observable maxima in the dynamical structure factor $S(\mathbf{q}, \omega)$ for $\mathbf{q} \neq 0$,

$$S(\mathbf{q}, \omega) = \frac{1}{V} \int_{-\infty}^{\infty} exp(i\omega t) < \rho_{\mathbf{q}}(t) \rho_{-\mathbf{q}}(0) >_0 \, dt, \tag{14}$$

$$\rho_{\mathbf{q}}(t) = \sum_{p} a^{+}_{\mathbf{p}-\mathbf{q}/2}(t) a_{\mathbf{p}+\mathbf{q}/2}(t), \tag{15}$$

where $\rho_{\mathbf{q}}(t)$ is the Fourier-component of the density operator in the Heisenberg representation. The dynamical structure factor $S(\mathbf{q}, \omega)$ (14) is directly related to [17] with the retarded density-density Green's function $\chi^R(\mathbf{q}, z)$ which is analytical in the upper semi-plane of the complex variable $z$ ($Imz > 0$),

$$S(\mathbf{q}, \omega) = -\frac{2\hbar}{1 - \exp(-\hbar\omega/T)} Im\chi^R(\mathbf{q}, \omega + i0), \tag{16}$$

$$\chi^R(\mathbf{q}, z) = -\frac{i}{\hbar V} \int_0^\infty dt \, \exp(izt) < [\rho_{\mathbf{q}}(t) \rho_{-\mathbf{q}}(0)] >_0 = \frac{1}{V} << \rho_{\mathbf{q}} \mid \rho_{-\mathbf{q}} >>_z, \tag{17}$$

The definitions (16),(17) have to be taken in the thermodynamic limit, where

$$\lim_{T \to 0} < N_0 >_0 = < N >_0 = N. \tag{18}$$

According to Eq. (14), the retarded function $\chi^R(q, z)$ can be represented in the form

$$\chi^R(\mathbf{q}, z) = \frac{1}{V} \sum_{p} F(\mathbf{p}, \mathbf{q}, z), \;\; F(\mathbf{p}, \mathbf{q}, z) = \langle\langle \, a^{+}_{\mathbf{p}-\mathbf{q}/2} a_{\mathbf{p}+\mathbf{q}/2} \mid \rho_{-q} \rangle\rangle_z \tag{19}$$

The equation of motion for the function $F(\mathbf{p}, \mathbf{q}, z)$ with the Hamiltonian $H_0$, determined by (Eq. (7)), can be written in the form

$$\left(\hbar z + \varepsilon_{\mathbf{p}-\mathbf{q}/\mathbf{2}} - \varepsilon_{\mathbf{p}+\mathbf{q}/\mathbf{2}}\right) F(\mathbf{p}, \mathbf{q}, z) = f_{\mathbf{p}-\mathbf{q}/2} - f_{\mathbf{p}+\mathbf{q}/2} -$$
$$\frac{1}{V} \sum_{k} u(k) \sum_{p_1} \langle\langle \, (a^{+}_{\mathbf{p}+\mathbf{k}-\mathbf{q}/2} a^{+}_{\mathbf{p_1}-\mathbf{k}/2} a_{\mathbf{p_1}+\mathbf{k}/\mathbf{2}} a_{\mathbf{p}+\mathbf{q}/2} - a^{+}_{\mathbf{p}-\mathbf{q}/2} a^{+}_{\mathbf{p_1}-\mathbf{k}/2} a_{\mathbf{p_1}+\mathbf{k}/2} a_{\mathbf{p}-\mathbf{k}+\mathbf{q}/2}) \mid \rho_{-q} \rangle\rangle_z \tag{20}$$

Here $f_{\mathbf{p}}$ is the one-particle distribution function over the momenta $\hbar\mathbf{p}$

$$f_{\mathbf{p}} = < a^{+}_{\mathbf{p}} a_{\mathbf{p}} >_0 \tag{21}$$

For a temperatures $T < T_0$, where $T_0$ is the condensation temperature, the one-particle distribution function $f_{\mathbf{p}}$ can be represented as [18]

$$f_{\mathbf{p}} = < N_0 > \delta_{\mathbf{p},0} + f^T_{\mathbf{p}} (1 - \delta_{\mathbf{p},0}), \tag{22}$$

where $N_0 = a^{+}_0 a_0$ is the operator of the number of particles with zero momentum ("condensate"), $f^T_{\mathbf{p}} = < a^{+}_{\mathbf{p}} a_{\mathbf{p}} >_0$ is the one-particle distribution function with non-zero momenta (the "overcondensate" states). Therefore,

$$n = n_0 + \frac{1}{V} \sum_{p \neq 0} f^T_{\mathbf{p}} = n_0 + \int \frac{d^3p}{(2\pi)^3} f^T_{\mathbf{p}}, \tag{23}$$

where $n_0 =< N_0 > /V$ is the average density of particles in the condensate. From (20) and (22), we find that the function $F(\mathbf{p},\mathbf{q},z)$ has singularities at $\mathbf{p}=\pm\mathbf{q}/2$. Therefore, the density-density function $\chi^R(q,z)$ (19) can be written as

$$\chi^R(q,z)=\frac{1}{V}\,F(\mathbf{q}/2,\mathbf{q},z)\,+\frac{1}{V}\,F(-\mathbf{q}/2,\mathbf{q},z)\,+\frac{1}{V}\sum_{p\neq\pm q/2}F^T(\mathbf{p},\mathbf{q},z) \tag{24}$$

The index $T$ means, that the respective function describes the "overcondensate" particles. The singularities (conditioned by the condensate) in this function are absent. Then in the last term in Eq. (24), we can change summation by integration over momenta. The functions $F(\pm\mathbf{q/2},\mathbf{q},z)$, extracted above, satisfy, according to Eqs. (20),(22), to the exact equations of motion

$$\begin{aligned}(\hbar z-\varepsilon_q)\;F(\mathbf{q}/2,\mathbf{q},z)=[\langle N_0\rangle-f^T_{\mathbf{q}}]-\frac{1}{V}\sum_{k\neq 0}u(k)\sum_{p_1}\langle\langle\,(a^+_{\mathbf{k}}a^+_{\mathbf{p_1}-\mathbf{k}/2}a_{\mathbf{p_1}+\mathbf{k}/2}a_{\mathbf{q}}-\\ a^+_0\,a^+_{\mathbf{p_1}-\mathbf{k}/2}a_{\mathbf{p_1}+\mathbf{k}/2}a_{\mathbf{q}-\mathbf{k}})\mid\rho_{-q}\rangle\rangle_z\end{aligned} \tag{25}$$

$$\begin{aligned}(\hbar z+\varepsilon_q)\;F(-\mathbf{q}/2,\mathbf{q},z)=-[\langle N_0\rangle-f^T_{\mathbf{q}}]-\frac{1}{V}\sum_{k\neq 0}u(k)\sum_{p_1}\langle\langle\,(a^+_{\mathbf{k}-\mathbf{q}}a^+_{\mathbf{p_1}-\mathbf{k}/2}a_{\mathbf{p_1}+\mathbf{k}/2}a_0-\\ a^+_{-\mathbf{q}}a^+_{\mathbf{p_1}-\mathbf{k}/2}a_{\mathbf{p_1}+\mathbf{k}/2}a_{-\mathbf{k}})\mid\rho_{-q}\rangle\rangle_z.\end{aligned} \tag{26}$$

Let us further consider the case of strongly degenerate gas, where $T\to 0$. To find for this case the main terms, following to the Bogolyubov's procedure, let us extract the terms on the right-hand sides of Eqs. (25),(26), which are determined by the maximum number of the operators $a^+_0$ and $a_0$. In the limit of strong degeneration, taking into account(18), we can omit the other terms in the course of calculation of $F$. Then, since $f^T_{\mathbf{q}}=f^T_{-\mathbf{q}}$ $(\mathbf{q}\neq 0)$, Eqs. (25),(26) take the form

$$(\hbar z\mp\varepsilon_q)\;F^{(0)}(\pm\mathbf{q}/2,\mathbf{q},z)=\pm[\langle N_0\rangle-f^T_{\mathbf{q}}]\pm\frac{1}{V}u(q)\langle\langle\,(a^+_0a^+_0a_{\mathbf{q}}a_0+a^+_0a^+_{-\mathbf{q}}a_0a_0)\mid\rho_{-q}\rangle\rangle_z \tag{27}$$

Eqs. (27) are exact at low temperature, since they define the functions $F^{(0)}(\pm\mathbf{q}/2,\mathbf{q},z)$. For calculation the Green's functions in the right side of Eqs. (27), it is necessary to make some approximations.

## IV. DETERMINATION OF THE COLLECTIVE EXCITATIONS

It should be emphasized that the Bogolyubov's approach, in which the operators $a^+_0$ and $a_0$ are considered as C-numbers, leads to violation of the exact relations (25) and (26). In

this approach (1), the main term $F^{(0)}$ of the function $F$ has the form [19]

$$F^{(0)}(\pm\mathbf{q}/2,\mathbf{q},z) = \langle N_0\rangle \,\langle\langle\, a_{\pm\mathbf{q}} \mid a^+_{\pm\mathbf{q}}\rangle\rangle_z \tag{28}$$

Respectively, instead of the exact equations for two-particle Green's functions (25) and (26), we obtain the equation of motion for one-particle Green's functions form (28). As is shown below, the equations of motion for the two-particle Green's functions without the approximation on the C-number representation of the operators $a_0^+ \;\; a_0$ are essentially different from the equations for the one-particle distribution functions. Therefore, for calculation of the Green's functions in the right part of (27), the assumption (1) cannot be used. At the same time, the idea of the C-number approximation for some operators seems, itself, very attractive. Below we apply this idea to the varsion alternative to the Bogolyubov assumption. According to (18), it is natural to accept that to calculate the Green's functions in the limit of strong degeneracy $T \to 0$, the C-number approximation has to be applied not to the operators $a_0^+ \;\; a_0$, but to the physical value - the operator of the number of particles in the "condensate" $N_0$

$$N_0 = \langle N_0\rangle. \tag{29}$$

For low temperatures $T$, the operator $N_0$ in the averages (under the angle brackets) can be changed by the total density operator $N$ (equal to $\rho_{q=0}$), which is the C-number in the canonical ensemble. Therefore, in the case under consideration, it can be removed from brackets. Therefore, for low temperatures, the proposed approximation seems asymptotically exact.

In this approximation from (27) directly follows

$$(\hbar z-\varepsilon_q)\; F^{(0)}(\mathbf{q}/2,\mathbf{q},z) = [\langle N_0\rangle - f^T_{\mathbf{q}}] + \frac{< N_0 >}{V} u(q)\{F^{(0)}(\mathbf{q}/2,\mathbf{q},z) + F^{(0)}(-\mathbf{q}/2,\mathbf{q},z)\} \tag{30}$$

$$(\hbar z+\varepsilon_q)\; F^{(0)}(-\mathbf{q}/2,\mathbf{q},z) = -[\langle N_0\rangle - f^T_{\mathbf{q}}] - \frac{< N_0 >}{V} u(q)\{F^{(0)}(\mathbf{q}/2,\mathbf{q},z) + F^{(0)}(-\mathbf{q}/2,\mathbf{q},z)\} \tag{31}$$

From Eqs. (30),(31), one can find the solutions for the functions $F(\pm q/2,q,z)$

$$F(\mathbf{q}/2,\mathbf{q},\hbar z) = \frac{[< N_0 > -f^T_{\mathbf{q}}](\hbar z+\varepsilon_q)}{(\hbar z)^2-(\hbar\omega(q))^2};\;\; F(-\mathbf{q}/2,\mathbf{q},z) = -\frac{[< N_0 > -f^T_{\mathbf{q}}](\hbar z-\varepsilon_q)}{(\hbar z)^2-(\hbar\omega(q))^2} \tag{32}$$

$$\hbar\omega(q) \equiv \sqrt{\varepsilon_q^2 + 2n_0 u(q)\varepsilon_q} \tag{33}$$

The relation (33) for the spectrum $\hbar\omega(q)$ corresponds exactly to the known Bogolyubov expression [1,2]. By substituting to (19), and taking into account that for the case of strong degeneration the contribution of the functions $F^T(\mathbf{p},\mathbf{q},z)$ is negligible, we obtain the expression for the main term $\chi^{(0)}(q,z)$ of the "density-density" Green function $\chi(q,z)$

$$\chi^{(0)}(q,z) = \frac{2n_0\varepsilon_q}{(\hbar z)^2 - (\hbar\omega(q))^2}\left\{1 - \frac{f_q^T}{< N_0 >}\right\} \tag{34}$$

As it is well known, the singularities of the function $\chi^R(q,z)$ determine the spectrum of collective excitations in the system. Therefore, under the assumption about C-number behavior of the operator $N_0$, we obtain the Bogolyubov's result for the spectrum of the collective excitations in the degenerated and weakly interacting Bose gas. However, the question on the braced term $f_{\mathbf{q}}^T/ < N_0 >$ in (34) remains open. The problem is in the behavior of the function $f_{\mathbf{q}}^{id}$ for the ideal Bose gas [20]

$$f_{\mathbf{q}}^{id} = \left\{\exp\left(\frac{\varepsilon(q)}{T}\right) - 1\right\}^{-1} \tag{35}$$

In the limit of small wave vectors $q$ the function $f_{\mathbf{q}}^{id}$ converges at non-zero temperatures (hereafter, $1/q^2$-divergence). Moreover,

$$\lim_{T\to 0}\lim_{q\to 0} f_q^{id} \neq \lim_{q\to 0}\lim_{T\to 0} f_q^{id} \tag{36}$$

The similar problem arises when (22) is used.

## V. ONE-PARTICLE EXCITATIONS

To calculate the distribution function $f_q^T$ for the "overcondensate" particles in Bose gas, let us consider the one-particle Green function $g^R(q,z)$

$$g^R(q,z) = \langle\langle\, a_{\mathbf{q}} \mid a_{\mathbf{q}}^+ \rangle\rangle_z, \quad \mathbf{q} \neq 0 \tag{37}$$

This function is directly connected with the distribution function $f_q^T$ by the relation [21]

$$f_q^T = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} g^<(q,\omega), \quad g^<(q,\omega) = -2\hbar\left\{\exp\left(\frac{\hbar\omega}{T}\right) - 1\right\}^{-1} Im\, g^R(q,\omega + i0) \tag{38}$$

The equation of motion for the Green function $g^R(q,z)$ for $q \neq 0$ reads

$$(\hbar z - \varepsilon_q)\; g^R(q,z) = 1 + \frac{1}{V}\sum_{k\neq 0} u(k) \sum_{p} \langle\langle\, a^+_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}} a_{\mathbf{q}+\mathbf{k}} \mid a^+_{\mathbf{q}} \rangle\rangle_z . \tag{39}$$

As for the "density-density" Green function, we consider the case of a strong degeneration and extract the terms with maximum quantity of the operators $a^+_0$ and $a_0$ in the right-hand side of Eq. (39). Then from Eq. (39), we find

$$(\hbar z - \varepsilon_q)\; g^R(q,z) = 1 + \frac{1}{V} u(q) \left\{ \langle\langle\, a^+_0 a_{\mathbf{q}} a_0 \mid a^+_{\mathbf{q}} \rangle\rangle_z + \langle\langle\, a^+_{-\mathbf{q}} a_0 a_0 \mid a^+_{\mathbf{q}} \rangle\rangle_z \right\} (1 - \delta_{q,0}), \tag{40}$$

In this case, it is assumed that the separated main term on the right-hand side of (40) corresponds to the case of the weak interaction. In the general case, when the interparticle interaction potential $u(q)$ is not small, in separating the terms with a maximum number of the operators $a^+_0$ and $a_0$ on the right-hand side of (40), it is necessary to write an infinite series on the perturbation theory on the interaction $u(q)$. It is clear that a similar situation occurs when using (27) instead of (26). Thereby, strictly speaking, the question whether the terms separated on the right-hand side of (40) are main from the viewpoint of the passage to the limit of the weak interaction remains open.

If we now apply the assumption (1) about the C-number representation of particle creation $a^+_0$ and annihilation $a_0$ operators at a zero momentum to the calculation of Green's functions on the right-hand side of (40), we come to the necessity of introducing the so-called "anomalous" Green's functions

$$g^R_{anom}(q,z) = \langle\langle\, a^+_{-\mathbf{q}} \mid a^+_{\mathbf{q}} \rangle\rangle_z, \qquad \mathbf{q} \neq 0, \tag{41}$$

Let us pay attention that the nonzero value of the anomalous function $g^R_{anom}$, strictly speaking, requires the influence of an external field of special form on the system under consideration. As a result, taking into account (1) and (41), relation (40) takes the form

$$(\hbar z - \varepsilon_q)\; g^R(q,z) = 1 + n_0 u(q) \left\{ g^R(q,z) + g^R_{anom}(q,z) \right\}. \tag{42}$$

Then we use the equation of motion for the anomalous function $g^R_{anom}(q,z)$ at $q \neq 0$ with Hamiltonian (7),

$$(\hbar z + \varepsilon_q)\; g^R_{anom}(q,z) = -\frac{1}{V}\sum_{k\neq 0}\sum_{p} u(k) \langle\langle\, a^+_{\mathbf{q}-\mathbf{k}} a^+_{\mathbf{p}-\mathbf{k}} a_p \mid a^+_{\mathbf{q}} \rangle\rangle_z , \tag{43}$$

Now, as in the above consideration, we separate terms with a maximum number of the operators $a_0^+$ and $a_0$ on the right-hand side of (43). Then, from (43), we obtain

$$(\hbar z + \varepsilon_q)\ g^R_{anom}(q,z) = -\frac{u(\mathbf{q})}{V}\left\{\langle\langle\, a_0^+ a_{-\mathbf{q}}^+ a_0 \mid a_{\mathbf{q}}^+ \rangle\rangle_z + \langle\langle\, a_0^+ a_0^+ a_{\mathbf{q}} \mid a_{\mathbf{q}}^+ \rangle\rangle_z \right\}, \tag{44}$$

Using the assumption (1) on the C-number representation of the particle creation $a_0^+$ and annihilation $a_0$ operators at a zero momentum in the calculation of Green's functions on the right-hand side of (44), we find

$$(\hbar z + \varepsilon_q)\ g^R_{anom}(q,z) = -n_0 u(q)\left\{g^R(q,z) + g^R_{anom}(q,z)\right\}. \tag{45}$$

It is evident that relations (42) and (45) form a set of algebraic equations in the functions $g^R(q,z)$ and $g^R_{anom}(q,z)$, from which it immediately follows that

$$g^R(q,z) = \frac{\hbar z + \varepsilon_q + n_0 u(q)}{(\hbar z)^2 - (\hbar\omega(q))^2}. \tag{46}$$

where the spectrum $\omega(q)$ is defined by relation (33). Thus, the poles of the one-particle Green's function $g^R(q,z)$ when using the assumption (1) about the C-number representation of the particle creation $a_0^+$ and annihilation $a_0$ operators at a zero momentum coincide with the poles of the "density-density" Green's function $\chi^R(q,z)$ (see (34)).

The above consideration is based on the assumption that it is sufficient to consider the terms with a maximum number of the operators $a_0^+$ and $a_0$ in the calculation of Green's functions at the temperature close to zero. In this case, it is supposed that the interparticle interaction is weak.

Let us now show that the result (46) can be obtained based on the consideration of the terms with a maximum number of the operators $a_0^+$ and $a_0$ without the use of anomalous Green's functions $g^R_{anom}(q,z)$. To this end, we write the equation of motion for the function $\langle\langle\, a_{-\mathbf{q}}^+ a_0 a_0 \mid a_{\mathbf{q}}^+ \rangle\rangle_z$ (see (40)),

$$(\hbar z + \varepsilon_q)\ \langle\langle\, a_{-\mathbf{q}}^+ a_0 a_0 \mid a_{\mathbf{q}}^+ \rangle\rangle_z = -\frac{1}{V}\sum_{k\neq 0}\sum_{p} u(k)\langle\langle\{a_{\mathbf{k-q}}^+ a_{\mathbf{p-k}}^+ a_p a_0 a_0 - a_{-\mathbf{q}}^+ a_{\mathbf{p-k}}^+ a_p a_{-\mathbf{k}} a_0 - a_{\mathbf{k-q}}^+ a_0 a_{\mathbf{p-k}}^+ a_p a_{-\mathbf{k}}\}$$

Then, on the right-hand side of (47), we separate the terms with a maximum number of the operators $a_0^+$ and $a_0$,

$$(\hbar z + \varepsilon_q)\ \langle\langle\, a_{-\mathbf{q}}^+ a_0 a_0 \mid a_{\mathbf{q}}^+ \rangle\rangle_z = -\frac{u(q)}{V}\left\{\langle\langle\, a_0^+ a_0^+ a_q a_0 a_0 \mid a_{\mathbf{q}}^+ \rangle\rangle_z + a_0^+ a_{-\mathbf{q}}^+ a_0 a_0 a_0 \mid a_{\mathbf{q}}^+ \rangle\rangle_z \right\}, \tag{48}$$

We now use the previously advanced assumption (29) that the operator of the number of "condensate" particles $N_0 = a_0^+ a_0$ is the C-number, rather than the operators $a_0^+$ and $a_0$. Then, from (48), it immediately follows that

$$(\hbar z + \varepsilon_q) \langle\langle a_{-\mathbf{q}}^+ a_0 a_0 \mid a_{\mathbf{q}}^+ \rangle\rangle_z = -n_0 u(q) \left\{ < N_0 > g^R(q,z) + \langle\langle a_{-\mathbf{q}}^+ a_0 a_0 \mid a_{\mathbf{q}}^+ \rangle\rangle_z \right\}, \quad (49)$$

or

$$\langle\langle a_{-\mathbf{q}}^+ a_0 a_0 \mid a_{\mathbf{q}}^+ \rangle\rangle_z = -\frac{n_0 u(q)}{(\hbar z)^2 + \epsilon(q) + n_0 u(q)} < N_0 > g^R(q,z), \quad (50)$$

**If we substitute relation (50) into (40), we again obtain expression for the one-particle Green's function $g^R(q,z)$.**

Thus, to obtain the known results for the spectra of excitations in the degenerate Bose gas, there is no need for the assumption (1) about the C-number representation of the particle creation $a_0^+$ and annihilation $a_0$ operators at a zero momentum and no need for putting into consideration anomalous Green's functions (41).

Let us now return to relation (34) for the "density-density" Green's function $\chi^R(q,z)$, from which it follows the necessity of calculating the one-particle distribution function $f_q^T$ for "overcondensate" states. Substituting the obtained expression (46) for the one-particle Green's function into (38), we find

$$f_q^T = \frac{\epsilon(q) + n_0 u(q)}{2\hbar\omega(q)} \coth\left(\frac{\hbar\omega(q)}{2T}\right) - \frac{1}{2}. \quad (51)$$

Let us pay attention that the dependence of the one-particle distribution function $f_q^T$ for "overcondensate" states on the order of limit transitions $T \to 0$ and $q \to 0$ (see (36)) again follows from (51). At the same time, to solve the problem of the braced term $f_q^T/\langle N_0 \rangle$ in relation (34), we can reason as follows. The minimum nonzero wave vector $q$ in a specified large but finite volume $V$ is proportional to $V^{-1/3}$. Therefore, in the thermodynamic limit $V \to \infty$, $N \to \infty$, $N/V = const$, the term $f_q^T/\langle N_0 \rangle$ can be considered to be zero provided that $n_0 = \langle N_0 \rangle / V \neq 0$, independently of the order of the limit transitions $T \to 0$ and $q \to 0$.

At first sight, the above results in the limit of the weak interaction completely solve the problem of the description of the weakly non-ideal degenerate Bose gas. However, it should be taken into account that, as follows from (23), (50), the number of particles in "overcondensate" states

$$\langle N_T \rangle = N - \langle N_0 \rangle = V \int d^3 q f_q^T \quad (52)$$

is nonzero even at zero temperature. Thus, in the limit $T \to 0$, the number of particles in the condensate $\langle N_0 \rangle \neq N$. Hence, the assumption about the C-number behavior of the operator $N_0 = a_0^+ a_0$, and even more so the operators $a_0^+$ and $a_0$ separately cannot be considered as grounded even in the limit of the weak interaction.

## VI. GAP AND THE SELF-CONSISTENT DISTRIBUTION FUNCTION

In this regard, let us return to the consideration of equation (40) under the assumption that we correctly considered the effects of the weak interparticle interaction. Then, taking into account the above consideration, we will not assume the C-number behavior of the operator $N_0 = a_0^+ a_0$ and the operators $a_0^+$ and $a_0$ separately. Then, from the viewpoint of the perturbation theory on the interparticle interaction, the second braced term on the right-hand side of (40) is the higher-order term in comparison with the first term; therefore, it can be neglected in the assumption of our interest. Let us pay attention that, from this point of view, the result (34) for the "density-density" Green's function remains valid.

Then, in the limit of a weak interparticle interaction, from Eq. (40) we obtain

$$g^R(q, z) = \frac{1}{\hbar z - E_q}, \tag{53}$$

The expression for the spectrum of the one-particle excitations $E_q$ is given by

$$E_q = \varepsilon_q + n_0 u(q), \tag{54}$$

From Eqs. (38),(52),(53), for the limit of strong degeneracy $T \ll T_0$, for the distribution function of the overcondensate particles, we obtain

$$f_q^T = \frac{1}{\exp(E_q/T) - 1}.. \tag{55}$$

Therefore, the function $f_q^T$ is finite for $q \to 0$. Moreover, in the limit of strong degeneracy $T \to 0$

$$f_q^T \to 0 \tag{56}$$

for arbitrary values of $q$, in contrast to the case of the ideal Bose gas. Therefore, the representation (22) for the one-particle distribution function $f_p$ is valid, the initial suggestions (2),(18) are satisfied and the used procedure of extraction of the main terms is self-consistent.

As is easily seen from (42),(43) for $T \to 0$ all particles in the considered approximation placed in the "condensate" with the momentum $p = 0$, in contrast to the Bogolybov's theory of weakly non-ideal homogeneous Bose gas (the effects of inhomogeneity will be considered in further publication). According to the accepted point of view, interaction between particles leads to the quantum depletion of the condensate, which means the appearance of the particles with $p \neq 0$ at $T = 0$. However, this statement is based on the assumption about coincidence of the excitation spectra for the one-particle Green function and dynamic structure factor. Only in this case the condensate density $n_0(T)$ can be calculated, based on the experimental data for $S(q, \omega)$ and the corresponding collective excitations in the spirit of the papers [22-24]. This coincidence takes place in the standard theory of weakly non-ideal Bose gas, which is constructed on the assumption about anomalous averages. This assumption is extended to liquid He II, where the approximate scheme for calculating $n_0$ is used. Therefore, the depletion of the Bose condensate is not an experimental fact, but is the consequence of applying of the theoretical concept of the identity of one-particle and collective spectra, obtained in the anomalous averages approach for weakly non-ideal Bose gas. It was shown above that *the theories based on C-number approximations for operators in the state with zero momentum are not self-consistent.*

In the developed theory the depletion is absent and, according to the above argumentation, there is no contradiction between this result and the existing experiments for the dynamical structure factor, since the latter cannot be applied to calculate of the particle distribution. In general, in a more elaborated theoretical approach the depletion can appear, however, even for such approach, the coincidence of the poles of one-particle Green function and the dynamical structure factor seems an exceptional occasion. Therefore, the Bose particle distribution should be found independently of the experiments on neutron scattering.

For the *self-consistent* approach under consideration the gap in the spectrum of one-particle excitations appears and is given by

$$\Delta = E_{q \to 0} = n_0 u(0). \tag{57}$$

A value of the gap is completely defined by the density of particles in the "condensate". The existence of the gap permits to extend essentially the applicability of the results obtained for $T \to 0$. It is obvious, that in many applications the condition $T \to 0$ is equivalent to the

condition $T \ll \Delta$.

Taking into account (44) we can rewrite Eq.(56) for the function $\chi^{(0)}(q,z)$ in the form

$$\chi^{(0)}(q,z) = \frac{2n_0\varepsilon_q}{(\hbar z)^2 - (\hbar\omega(q))^2} \tag{58}$$

On the basis of (58) almost all known results for the thermodynamical functions of the degenerate Bose gas can be reproduced, as is done in [9-11].

Therefore, in contrast with the approach, based on the C-number approximation for the operators $a_0^+$ and $a_0$ (or applying the C-number approximation for the operator $N_0$ in the direct perturbation theory in the presence of the condensate) we find that the spectra of the collective and one-particle excitations are different. Both spectra, as is easily seen, satisfy the Landau condition for superfluidity. For the one-particle spectrum the Landau condition is satisfied for the transitions between the "condensate" and the "overcondensate" state. The Landau condition is, naturally, violated for transitions between the "overcondensate" states.

## VII. CONCLUSIONS

Summarizing the above consideration, we can assert that the calculation of the Green's functions for the highly degenerate Bose gas on the basis of the C-number approximation for the operator $N_0$ and the straightforward perturbation theory allows the following conclusions.

(i) The problem of $1/q^2$ divergence, which arises for the ideal Bose gas, can be solved.

(ii) The system has two different branches of excitations, i.e., the single-particle and collective ones, both satisfying the Landau condition of superfluidity.

(iii) The single-particle excitation spectrum contains the gap in the region of small wave vectors, associated with the existence of the "condensate".

(iv) The spectrum of collective excitations corresponds to "phonon-roton" excitations observed in the experiments on inelastic neutron scattering [23,24,32]. This spectrum is the gapless mode which has no connection with the Goldstone theorem.

(v) There is no need for anomalous averages (quasi-averages) to describe the degenerate Bose gas.

In this connection, we emphasize that the Goldstone theorem [25] which justifies the gapless branch of excitations is itself the result of the breaking symmetry assumption. Therefore, it cannot be applied to the theory considered above, which is constructed without breaking symmetry assumption (e.g., without anomalous averages). In our paper, we showed that the breaking symmetry is not necessary (in this connection see also [26-29], where the various approaches to avoid the breaking symmetry have been developed) and at the same time the Bogolyubov result for the collective phonon-roton mode is completely reproduced. In [30,31], the analysis of the applicability of the relativistic Goldstone theorem to the non-relativistic statistical theory has been done.

It should also be mentioned that Hugenholtz and Pines, constructing the diagram technique for the degenerated Bose gas in [3], reformulated the problem at the beginning, by changing, according to Bogolyubov, the operators $a_0^+$ and $a_0$ by C-numbers. On this way, they could apply almost automatically the quantum field theory methods to the study of the Bose gas with "condensate". In particular, it was shown that, for the C-number representation of the operators $a_0^+$ and $a_0$, the gap in excitations associated with the one-particle Green's function cannot exist. This result is also associated with the anomalous averages assumption.

In the case at hand, it should be noted that this assumption has no any fundamental basis, is not used for the ideal Bose gas and, as we showed above, can be avoided for the non-ideal Bose gas. Historically, the assumption of breaking symmetry in the Bose gas theory played a crucial role and was very useful, but in fact it is not necessary.

The proposed theory does not contradict the fundamental physical laws; the known experiments on the collective mode can be qualitatively described since this mode is the same as that in the existing theory based on the breaking symmetry assumption. The Landau criterion of superfluidity is also fulfilled.

Therefore, the interparticle interaction in Bose systems leads not only to the drastic difference (in comparison with the ideal Bose gas) in the structure of collective excitations, which are described by the "density-density" Green's function, but also to the crucial change in the distribution function of single-particle excitations and in the single-particle excitation spectrum of for "overcodensate" particles.

Based on the results obtained, the special diagram technique can be developed, similar to [19], but with the use of the C-number approximation for the operator $N_0$.

The principal difference between the results of this study and the results of the "traditional" C-number approximation for the operators $a_0^+$ and $a_0$ (as well as for the operator $N_0$) is the existence of the gap in the spectrum of single-particle excitations. The above analysis shows that this gap cannot manifest itself in the experiments on inelastic neutron scattering in superfluid helium (see, e.g., [32-34]). However, such possibility cannot be excluded [16] in the experiments on Raman light scattering. Moreover, in [35-38], where such experiments are described, there is a direct indication of the existence of the gap.

Probably, the gap for single-particle excitations can manifest itself in the experiments on the density profile in trapped Bose gases [39]; however, this question cannot be considered within the present paper and has to be studied separately.

**Acknowledgment**

The authors thank Yu. A. Kuharenko and A.G. Zagorodny for the helpful discussions. The authors express gratitude to the Netherlands Organization for Scientific Research (NWO) for support of their investigations on the problems of statistical physics.

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[1] N.N. Bogolubov, J.Phys.USSR **11** (1947) 23

[2] N.N. Bogolubov, Lectures on quantum statistics, Kyiv "Radianska Shkola" 1949; Selected works v.2 (1969)

[3] N.M.Hugenholtz and D.Pines, Phys.Rev. **116** (1959) 489

[4] A.A. Abrikosov, L.P. Gorkov, I.E. Dzyaloshinskii, Quantum Field Theoretical Methods in Statistical Physics, 2ed., Pergamon, (1965)

[5] W.H.Bassichis and L.L.Foldy, Phys.Rev. **133A** (1964) 435

[6] H.Stolz, Physica **A86** (1977) 11

[7] S.A. Trigger and P.P.J.M.Schram, Physica **B 228** (1996) 107

[8] I.M. Yurin, The Question Wheather an Energy Gap Does Exist in He II or Not, Archiv, [ quant-ph.] 0310115 v.1 (2003)

[9] V.B. Bobrov, S.A. Trigger, Physica **A 170** (1990) 261

[10] V.B. Bobrov, Yu.P. Vlasov, S.A. Trigger, JETP 75, 56 (1992)

[11] V.B. Bobrov, S.A. Trigger, P.P.J.M. Schram, JETP 80, 853 (1995)

[12] P. Szèpfalusi, I Kondor, Annals of Physics **82**, (1974) 1

[13] I.M. Yurin, S.A. Tigger, The Possibility of Existence of an Energy Gap in the One-Particle Excitation Spectrum of HeII, ArXiv: [cond-mat.supr-con] 0906.0755 v2 (2009)

[14] C.-H. Zhang, H.A. Fertig, Phys. Rev. **A74** (2006) 023613

[15] S.A. Trigger, I.M. Yurin, Extensions of the Bogolyubov's Theory of Weakly Non-ideal Bose Gas, Modern problems of theoretical and mathematical physics, Bogolyubov Kiev conference, Book of abstracts, p. 36 (2009)

[16] V.B. Bobrov, S.A. Trigger, I.M. Yurin, Phys. Letters **A 374** (2010) 1938; V.B. Bobrov, S.A. Trigger, Collective Excitations and the Gap in the Spectrum of the Degenerated Bose Gas, ArXiv: [cond-mat.supr-con.] 0910.3455 v1 (2009)

[17] E.M. Lifshitz, L.P. Pitaevsky, Statistical Physics, part 2 Thoery of condensed matter, Nauka, Moscow (1978) [in Russian]

[18] Yu.L. Klimontovich, V.P. Silin, JETP **23** (1952) 151

[19] Yu.A.Tserkovnikov, Theor.Math.Phys. 4, 720 (1970)

[20] R. Balescu, Equilibrium and nonequilibrium statistical mechanics, v.1, J. Wiley and Sons, Inc. (1975)

[21] L.P. Kadanoff, G. Baym, Quantum statistical mechanics, W.A. Benjamin, Inc. N.-Y. (1962)

[22] V. F. Sears, E. C. Svensson, P. Martel, and A. D. B. Woods, Phys. Rev. Lett. **49** (1982) 279

[23] H. R. Glyde, R. T. Azuah and W. G. Stirling, Phys. Rev. B, **62** (2000) 14337

[24] H. R. Glyde, S. O. Diallo, R. T. Azuah, O. Kirichek, and J. W. Taylor, H. R. Glyde, R. T. Azuah and W. G. Stirling, Phys. Rev. B, **83** (2011) 100503 (R)

[25] J.Goldstone, *Nuovo Cim.* **19** (1961) 154; J.Goldstone, A.Solam and S.Weinberg, *Phys.Rev.***127** (1962) 965

[26] M.D. Girardeau, R. Arnowitt, Phys. Rev. 133 (1959) 755

[27] C. W. Gardiner, PHys. Rev. A 56, (1997) 1414

[28] M. D. Girardeau, Phys. Rev. A 58 (1998) 775

[29] Y. Castin, R. Dum, Phys. Rev. A, 57, (1998) 3008

[30] R. V. Lange, Goldstone Theorem in Non-relativistic Theories, *Phys. Rev. Lett.*, **14** (1965) 3

[31] N. M. Hugenholtz, Rep. Prog. Phys. **28** (1965) 201

[32] R.A.Cowley and A.D.B.Woods, Can.J.Phys. **49** (1971) 177

[33] N.M. Blagoveshchensky, E.B. Dokukin, J.A. Kozlov, V.A. Parfenov, Pis'ma v ZhETF **31** (1980) 7 (Sov.Phys. JETP Letters)

[34] R. J. Donnelly, J. A. Donnelly, and R. N. Hills, J. Low Temp. Phys. **44** (1981) 471

[35] T.J.Greytak, R. Woerner, J.Yan, R.Benjamin, Phys. Rev. Lett. **25** (1970) 1547

[36] K. Ohbayashi, M. Udagawa, Phys. Rev. B, **31** (1985) 1324

[37] K. Ohbayashi, M. Udagawa, N. Ogita, Phys. Rev. B, **58** (1998) 3351

[38] M. Shay, O. Pelleg, E. Polturak, S. G. Lipson, Phys. Rev. **B 75**, (2007) 054516

[39] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, S. Stringari, Reviews of Modern Physics **71** (1999) 3